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Convergence of the time-invariant Riccati differential equation and LQ-problem: mechanisms of attraction

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Convergence of the time-invariant Riccati differential equation and LQ-problem: mechanisms of attraction

FRANK M. CALLIER†, JOSEPH WINKIN†
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The nature of the attraction of the solution of the time-invariant matrix Riccati differential equation towards the stabilizing solution of the algebraic Riccati equation is studied. This is done on an explicit formula for the solution when the system is stabilizable and the hamiltonian matrix has no eigenvalues on the imaginary axis. Various aspects of this convergence are analysed by displaying explicit mechanisms of attraction, and connections are made with the literature. The analysis ultimately shows the exponential nature of the convergence of the solution of the Riccati differential equation and of the related finite horizon LQ-optimal state and control trajectories as the horizon recedes. Computable characteristics are given which can be used to estimate the quality of approximating the solution of a large finite-horizon LQ problem by the solution of an infinite-horizon LQ problem.

1. Introduction

In the past three decades, the matrix Riccati equation has received a great deal of attention (e.g. Bittanti *et al.* 1991). A particular question considered is the asymptotic behaviour of the solution of the Riccati differential equation (RDE) related to the linear-quadratic (LQ) optimal control problem, (see for example Kwakernaak and Sivan 1972, Rodriguez-Canabal 1973, Faurre *et al.* 1979, Callier and Willems 1981, Brockett 1970, Anderson and Moore 1989, Shayman 1986, Kailath and Ljung 1976, Callier and Desoer 1991). The present paper studies properties of the convergence of the solution of the time-invariant RDE to the stabilizing solution of the algebraic Riccati equation (ARE) on an *explicit formula*. The results are obtained by displaying explicit mechanisms of attraction and the analysis ultimately shows the *exponential* nature of the convergence of the solution of the RDE and of the associated finite-horizon optimal state and control trajectories as the horizon recedes. As a byproduct, estimates are obtained of the quality of approximating a large finite-horizon LQ problem by an infinite-horizon LQ problem.

The following notations and definitions are used throughout. For a square matrix $A \in \mathbb{R}^{n \times n}$; $L^-(A)$, $L^0(A)$ and $L^+(A)$ denote the A-invariant subspaces of $\mathbb{R}^n$ spanned by a basis of (generalized) eigenvectors corresponding to the eigenvalues λ of A with, respectively, negative, zero and positive real parts. For a matrix A, $\mathcal{N}(A)$ and $\mathcal{R}(A)$ denote respectively the null space and the range, and A^* the hermitian transpose. For any hermitian matrix $A = A^*$, for any subspace V of $\mathbb{R}^n$, '$A \geq 0$, (respectively > 0) on V' means that A is positive semi-definite (respectively positive definite) on V, i.e. $x^*Ax \geq 0$ (respectively

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> 0), for every non-zero $x \in V$. For a vector x and a matrix A, $\|x\|$ and $\|A\|$ denote respectively the euclidean norm of x and the induced euclidean norm, i.e. the largest singular value of A.

We also need the following standard concepts (e.g. Callier and Willems 1981, Callier and Desoer 1991). For a matrix pair (A, B), with $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$, $C(A, B) := \mathcal{R}([B \; AB \; \dots \; A^{n-1}B])$ is the controllable subspace of $\mathbb{R}^n$ (defined by (A, B)) and $S(A, B)$ is the stabilizable subspace of $\mathbb{R}^n$, namely

$$S(A, B) := L^-(A) + C(A, B)$$

For a matrix pair (C, A), with $C \in \mathbb{R}^{p\times n}$ and $A \in \mathbb{R}^{n\times n}$, $NO(C, A) := \bigcap_{i=0}^{n-1}\mathcal{N}(CA^i)$ is the unobservable subspace of $\mathbb{R}^n$ and $ND(C, A)$ denotes the undetectable subspace of $\mathbb{R}^n$, namely

$$ND(C, A) := NO(C, A) \cap (L^0(A) + L^+(A))$$

(A, B) is said to be controllable (respectively stabilizable) iff $C(A, B) = \mathbb{R}^n$, (respectively $S(A, B) = \mathbb{R}^n$). (C, A) is said to be observable (respectively detectable) iff $NO(C, A) = \{0\}$, (respectively $ND(C, A) = \{0\}$).

For reasons of motivation we briefly recapitulate certain facts. The standard finite-horizon LQ problem can be stated as follows: consider the linear time-invariant state differential equation

$$\dot{x}(t) = Ax(t) + Bu(t) \quad t \geq 0 \tag{1}$$

with initial condition

$$x(0) = x_0 \in \mathbb{R}^n \tag{2}$$

and control constraint

$$u(\cdot) \in \mathcal{U} \tag{3 a}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and the set of admissible controls is given by

$$\mathcal{U} = \{u(\cdot): u(\cdot): \mathbb{R}_+ \to \mathbb{R}^m \text{ is continuous}\} \tag{3 b}$$

Let

$$S = S^* \geq 0 \tag{4}$$

be a symmetric positive semi-definite (terminal state penalty) matrix in $\mathbb{R}^{n\times n}$ and let $C \in \mathbb{R}^{p\times n}$. Consider the finite horizon quadratic cost

$$V(x_0, t_1, u, S) = \int_0^{t_1}(\|Cx(t)\|^2 + \|u(t)\|^2)\,dt + x(t_1)^*Sx(t_1) \tag{5}$$

Problem (LQ): For any fixed horizon $t_1 > 0$, find an optimal control $u(\cdot) \in \mathcal{U}$ which minimizes the cost $V(x_0, t_1, u, S)$, (1)–(5), for an arbitrarily fixed initial state $x_0 \in \mathbb{R}^n$. □

The solution of problem (LQ) is based on the solution $P(t, t_1, S) = P(t, t_1, S)^* \geq 0$ of the RDE given by

$$-\dot{P}(t) = A^*P(t) + P(t)A - P(t)BB^*P(t) + C^*C \quad t \leq t_1 \tag{RDE}$$

for $P(t_1) = S$.

Proposition 1—(Kwakernaak and Sivan 1972, Theorem 3.4, p. 218) **Solution of Problem (LQ):** *The solution of problem (LQ) is such that*

(*a*) *the optimal cost* $V^0(x_0, t_1, S)$ *is given by the non-negative quadratic form*

$$V^0(x_0, t_1, S) = x_0^* P(0, t_1, S) x_0 \tag{6}$$

(*b*) *the optimal control* $u(\cdot) \in \mathcal{U}$ *is given by the optimal state feedback*

$$u(t) = -B^* P(t, t_1, S) x(t) \quad t \in [0, t_1] \tag{7}$$

where by the substitution of $u(\cdot)$ *in* (1), *the optimal state trajectory* $x(\cdot)$ *on* $[0, t_1]$ *satisfies the closed loop differential equation*

$$\dot{x}(t) = [A - BB^* P(t, t_1, S)] x(t), \quad t \in [0, t_1] \tag{8 a}$$

$$x(0) = x_0 \in \mathbb{R}^n \tag{8 b}$$

In this paper it is assumed that

$$\text{(A, B) is stabilizable and H has no eigenvalues on the imaginary axis} \tag{A}$$

where H is the hamiltonian matrix given by

$$H = \begin{bmatrix} A & -BB^* \\ -C^*C & -A^* \end{bmatrix} \tag{9}$$

Note (Kučera 1972, Molinari 1977), that (A) holds if

$$\text{(A, B) is stabilizable and (C, A) is detectable.} \tag{B}$$

Moreover, (Kucera 1972, Molinari 1977, p. 354, and the references therein) assumption (A) is necessary and sufficient in order that the algebraic Riccati equation given by

$$A^*P + PA - PBB^*P + C^*C = 0 \tag{ARE}$$

has a unique symmetric positive semi-definite solution $P_+ = P_+^* \geq 0$ that is stabilizing, i.e. such that the closed loop A-matrix given by

$$A_+ = A - BB^* P_+ \tag{10 a}$$

is exponentially (exp.) stable, or equivalently

$$\text{the eigenvalues of A_+ have negative real parts.} \tag{10 b}$$

Assume now that, for some $S = S^* \geq 0$ and for every fixed t with $t \leq t_1$

$$\lim_{t_1 \to \infty} P(t, t_1, S) = P_+ \tag{11}$$

(note that (11) holds for every $S = S^* \geq 0$ if assumption (B) holds, (e.g. Kwakernaak and Sivan 1972, Theorem 3.7; Kailath and Ljung 1976)). It is then possible to consider infinite horizon costs depending upon $S = S^* \geq 0$ given by

$$V(x_0, \infty, u, S) = \lim_{t_1 \to \infty} V(x_0, t_1, u, S) \tag{12}$$

where $V(x_0, t_1, u, S)$ is the finite horizon cost (1)–(5). This leads to the following infinite-horizon LQ-optimal control problem

Problem (LQ)$^\infty$: Find an optimal control $u(\cdot) \in \mathcal{U}$ which minimizes the cost $V(x_0, \infty, u, S)$, given by (1)–(5) and (12), for an arbitrarily fixed initial state $x_0 \in \mathbb{R}^n$. □

By slightly modifying the proof of Callier and Desoer (1991, Theorem 10.4.91, p. 42) we have now the following proposition.

Proposition 2—Solution of (LQ)$^\infty$: *Let assumption* (A) *hold. Let $P_+ = P_+^* \geq 0$ be the unique stabilizing solution of the ARE. Let A_+ be the corresponding closed loop matrix given by (10). Let $S = S^* \geq 0$ be such that (11) holds. Then the solution of problem $(LQ)^\infty$ is such that*

(*a*) *the optimal cost $V^o(x_0, \infty, S)$ is given by the non-negative quadratic form*

$$V^o(x_0, \infty, S) = x_0^* P_+ x_0 \tag{13}$$

(*b*) *the optimal control $u(\cdot) \in \mathcal{U}$ is given by the optimal state feeback*

$$u(t) = -B^* P_+ x(t) \quad \text{on } t \geq 0 \tag{14}$$

where, by the substitution of $u(\cdot)$ in (1), *the optimal state trajectory $x(\cdot)$ on $t \geq 0$ satisfies the closed loop exp. stable differential equation*

$$\dot{x}(t) = A_+ x(t), \quad \textit{with } x(0) = x_0 \in \mathbb{R}^n \tag{15}$$

Observe that Proposition 2 holds for every $S = S^* \geq 0$ if (B) holds. However, even in that case, the qualitative properties of the attraction of $P(t, t_1, S)$ to P_+ as $t_1 \to \infty$ and hence those of the attraction of the optimal state and control trajectories of Proposition 1 to those of Proposition 2 are dependent upon S. It is the purpose of the present paper to display the role of S and related mechanisms of attraction under assumption (A). This is done as follows.

An explicit formula for the difference $P(t, t_1, S) - P_+$ is obtained in Theorem 1: it is an obvious generalization of earlier results. Although convergence results of $P(t, t_1, S)$ to P_+ are essentially known (e.g. Kwakernaak and Sivan 1972, Callier and Willems 1981), they were not derived using this formula. We study therefore its specific mechanism of attraction: Theorem 2 yields three equivalent explicit criteria for the convergence of $P(t, t_1, S)$ to P_+ for a given terminal state penalty matrix S: one of them is exactly the condition (32) of Callier and Willems (1981, part IV) and a second one generalizes another criterion of Faurre *et al.* (1979, § 5.2): one hereby uses a system theoretic interpretation using the undetectable subspace of (C, A); finally in Corollary 1 it is shown that attraction holds for every $S = S^* \geq 0$ if the stronger assumption (B) holds. The next results show that the explicit formula for $P(t, t_1, S) - P_+$ is very useful when $P(t, t_1, S)$ converges to P_+ as $t_1 \to \infty$. Theorem 3 shows that this convergence is *exponential* with *computable characteristics*; moreover, the Riccati differential equation governing $P(t, t_1, S) - P_+$ is shown to be *linearizable* through a homographic (i.e. linear fractional) transformation: more precisely, the solution of the former differential equation can be expressed, by means of this (invertible) transformation, in terms of the solution of an exp. stable linear matrix differential equation, which is asymptotically (for $t_1 \to \infty$) the dominant part of $P(t, t_1, S) - P_+$. Finally, we show in Theorem 4 that, as the horizon recedes, the optimal state and control trajectories of Proposition 1 are *exponentially* attracted to those of Proposition 2 in any $L_p[0, t_1]$-norm with $p \in [1, \infty]$.

2. Explicit formula for the solution of the RDE

We begin by stating two standard lemmas needed below.

Lemma 1—(Coppel 1974, pp. 274–275, Callier and Desoer 1991, pp. 35–37): *Consider any fixed horizon* $t_1 > 0$ *and any terminal state penalty matrix* $S = S^* \geqslant 0$. *Consider the hamiltonian matrix* $H \in \mathbb{R}^{2n\times 2n}$ *defined by (9). Let* $X(\cdot)$ *and* $Y(\cdot)$: $(-\infty, t_1] \to \mathbb{R}^{n\times n}$ *be the solutions of the backwards hamiltonian matrix differential equation*

$$\begin{bmatrix} \dot{X}(t) \\ \dot{Y}(t) \end{bmatrix} = H\begin{bmatrix} X(t) \\ Y(t) \end{bmatrix} \quad \text{with } X(t_1) = I \text{ and } Y(t_1) = S \tag{16}$$

Then

(*a*)

$$\text{for any } t \leqslant t_1,\ X(t) \text{ is non-singular} \tag{17}$$

(*b*) *the solution* $P(t, t_1, S)$ *of the RDE satisfies*

$$P(t, t_1, S) = Y(t)X(t)^{-1}, \quad \text{for all } t \leqslant t_1 \tag{18}$$

(*c*) *the optimal state trajectory* $x(\cdot)$, (8), *of problem (LQ) is given by*

$$x(t) = X(t)X(0)^{-1}x_0, \text{ for all } t \in [0, t_1] \tag{19}$$

Lemma 2: *Let assumption* (A) *hold. Let* $P_+ = P_+^* \geqslant 0$ *be the unique stabilizing solution of the ARE. Let* A_+ *be the corresponding closed loop matrix given by (10). Consider the closed loop reachability grammian on* $[0, \tau]$ *with* $\tau := t_1 - t$, *given by*

$$W(\tau) := \int_0^\tau \exp(A_+ t)BB^* \exp(A_+^* t)\,\mathrm{d}t \tag{20}$$

and its corresponding grammian on $[0, \infty)$, *given by*

$$W := \lim_{\tau\to\infty} W(\tau) = \int_0^\infty \exp(A_+ t)BB^* \exp(A_+^* t)\,\mathrm{d}t \geqslant 0. \tag{21}$$

Then

(*a*) *W is the unique solution of the closed-loop Lyapunov equation*

$$WA_+^* + A_+W + BB^* = 0 \tag{22}$$

(*b*) *as* $\tau \to \infty$, $W(\tau)$ *has an exp. converging evolution dictated by*

$$W(\tau) = W - \exp(A_+\tau)W\exp(A_+^*\tau) \tag{23}$$

Comments 1:

(α) The proof of Lemma 2 is standard (e.g. Brockett 1970, Theorem 3, p. 61).

(β) If, in Lemma 2, (A, B) is assumed to be controllable, then W, (20)–(21), is recognized in the literature as the 'inverse gap', i.e. $W = (P_+ - P_-)^{-1} > 0$, where $P_+ = P_+^* \geqslant 0$ and $P_- = P_-^* \leqslant 0$ are respectively the stabilizing and antistabilizing solutions of the ARE (see for example Molinari 1977, proof of Theorem 6; Willems 1971, Remark 15 and proof of Lemma 8). □

Since the LQ problem is time-invariant, the solution $P(t, t_1, S)$ of the RDE is, from now on, denoted by $P(\tau)$, where $\tau := t_1 - t$ as in Lemma 2. By Lemmas 1 and 2 we obtain the following as in Sorine and Winternitz (1985, Section II).

Theorem 1—Explicit formula for the solution of the RDE: *Let* (A) *hold. Let $P_+ = P_+^* \geqslant 0$ be the unique stabilizing solution of the ARE and let A_+ be the corresponding closed loop matrix given by (10). Let $W = W^* \geqslant 0$ be the unique solution, (20), of the closed loop Lyapunov equation (22) and let $W(\tau)$ be the reachability grammian given by (20), (23). Finally let $P(\tau)$ be the solution of the RDE, with $\tau = t_1 - t$. Then, for any $S = S^* \geqslant 0$*

$$\Delta P(\tau) := P(\tau) - P_+ = \exp(A_+^* \tau)\tilde{S}(\tau)\exp(A_+ \tau), \quad \tau \geqslant 0 \tag{24 a}$$

where the symmetric matrix function $\tilde{S}(.)$, given by

$$\tilde{S}(\tau) := (S - P_+)[I + W(\tau)(S - P_+)]^{-1} \tag{24 b}$$

is well defined on $\mathbb{R}_+$ and where

$$W(\tau) = W - \exp(A_+ \tau) W \exp(A_+^* \tau) \tag{24 c}$$

Comments 2:

(α) Theorem 1 follows directly from Lemma 1(a)–(b) by applying the similarity transformation

$$\tilde{H} := T^{-1} H T, \quad \text{where } T := \begin{bmatrix} I & 0 \\ P_+ & I \end{bmatrix}$$

to the hamiltonian matrix H given by (9).

(β) Formula (24) is a special case of Sorine and Winternitz (1985, Formula (16)) and Anderson and Moore (1989, Appendix E4) and a dual version of Kailath and Ljung (1976, Formula (12)), Sasagawa (1982, Corollary 1), Rusnak (1988, Formula (2)): it uses only the knowledge of P_+ and W. It is similar to other existing formulae, (e.g. Brockett 1970, p. 150, Rodriguez-Canabal 1973, Equation (3.3), Faurre *et al.* 1979, equation (5.11)); those are valid, however, under more restrictive conditions, the most important one being that (A, B) be controllable, from which P_+ and P_- can be used explicitly.

(γ) Formula (24) above for $\Delta P(\tau)$ would be the solution of a linear matrix differential equation if the symmetric matrix function $\tilde{S}(\tau)$ given by (24 b) was constant. In that case, with $\tau = t_1 - t$, $\tilde{S}(\tau) = \tilde{S}(0)$ would be the terminal condition of $\Delta P(\tau)$. However, as will be made clear in § 4 below, $\tilde{S}(\tau)$ is a time-varying decreasing function. For these reasons we shall call the symmetric matrix function $\tilde{S}(\tau)$ in (24 b) the *sliding terminal condition* of $\Delta P(\tau)$: it plays a crucial role in § 4. □

3. Convergence of the solution of the RDE

The purpose of this section is to use formula (24) to obtain conditions of attraction of $P(\tau)$ towards P_+, that are well related to the literature. We start with a brief lemma.

Lemma 3: *Let* (A) *hold. Let* $S = S^* \geq 0$ *be given. Then*

$$\Delta P(\tau) := P(\tau) - P_+ \to 0 \quad as\ \tau \to \infty \tag{25}$$

iff

$$I + W(S - P_+) \text{ is non-singular} \tag{26}$$

Proof

Sufficiency. By (24 b), (21) and (26), $\tilde{S}(\tau)$ tends to $\tilde{S} := (S - P_+)[I + W(S - P_+)]^{-1}$ as $\tau \to \infty$. Hence (25) holds by (24 a) and (10).

Necessity. It follows from (24 b–c) that

$$\tilde{S}(\tau)[I + W(S - P_+) - \exp(A_+\tau)W\exp(A_+^*\tau)(S - P_+)] = S - P_+ \tag{27}$$

Assume for a contradiction that there exists an $x \in \mathbb{R}^n$ such that

$$x \neq 0 \text{ and } [I + W(S - P_+)]x = 0 \tag{28}$$

Hence, by (27) and (24 a),

$$-W\Delta P(\tau)W\exp(A_+^*\tau)(S - P_+)x = W\exp(A_+^*\tau)(S - P_+)x \tag{29}$$

Now, by (28), $W(S - P_+)x \neq 0$; therefore the holomorphic vector-valued function on the right-hand side of (29) is not identically zero on $\tau \geq 0$. Thus, the function $\tau \mapsto \|W\exp(A_+^*\tau)(S - P_+)x\|$ has only isolated zeros on $\tau \geq 0$. It follows, by (29) and by the continuity of the function $\tau \mapsto \|W\Delta P(\tau)\|$, that

$$1 \leq \|W\Delta P(\tau)\| \leq \|W\| \cdot \|\Delta P(\tau)\| \quad \text{on } \tau \geq 0$$

Hence, (25) leads to a contradiction. □

In the following, the symmetric positive semi-definite matrix $W^\#$ denotes the Moore-Penrose *generalized* (or least-squares) *inverse* of the reachability grammian W, (21), (see for example Noble and Daniel 1977, pp. 339–341), such that $WW^\#$ is the orthogonal projection onto $\mathcal{R}(W)$ and $WW^\# = W^\#W$, from which

$$W^\#Wx = WW^\#x = x \quad \text{for all } x \in \mathcal{R}(W) \tag{30}$$

Recall also that for all $\tau > 0$,

$$\mathcal{R}(W) = \mathcal{R}(W(\tau)) = C(A_+, B) = C(A, B) \tag{31}$$

Criterion (26) above needs to be explained. The next result enables its system theoretic interpretation in (35) below. We are inspired here by Molinari (1977), Willems (1971).

Lemma 4: *Let* (A) *hold. Then*

(a) *for any* $x_0 \in \mathcal{R}(W)$

$$x_0^*(W^\# - P_+)x_0 = \int_{-\infty}^{0} (\|Cx(t)\|^2 + \|u(t)\|^2)\,dt \tag{32 a}$$

where, on $t \leq 0$

$$\dot{x}(t) = Ax(t) + Bu(t),\ x(0) = x_0 \tag{32 b}$$

with

$$u(t) = -B^*P_+x(t) + v(t) \tag{32 c}$$

and

$$v(t) = B^* \exp(-A_+^* t) W^\# x_0 \tag{32 d}$$

Moreover, under these conditions

$$x(t) = W \exp(-A_+^* t) W^\# x_0 \quad \text{on } t \leqslant 0 \tag{32 e}$$

(b) $P_+ - W^\#$ *is the negative semi-definite and anti-stabilizing solution on* $\mathcal{R}(W)$ *of the controllable restriction of the ARE, i.e. with* $P = P_+ - W^\#$

$$y^*[A^*P + PA - PBB^*P + C^*C]x = 0 \quad \textit{for all } x,\, y \in \mathcal{R}(W) \tag{33}$$

(c)

$$\mathcal{N}(I - WP_+) = \mathcal{N}(P_+ - W^\#) \cap \mathcal{R}(W) = ND(C, A) \tag{34}$$

(d) *for any* $S = S^* \geqslant 0$

$$\mathcal{N}(I + W(S - P_+)) = \mathcal{N}(S) \cap ND(C, A) \tag{35}$$

Comments 3:

(α) The results in (32) can be shown to describe the solution of the dual negative-time LQ-optimal reachability problem, namely for any $x_0 \in \mathcal{R}(W)$, find a control $u(\cdot)$ which minimizes the cost functional

$$J(x_0, u(\cdot)) = \int_{-\infty}^{0} (\|Cx(t)\|^2 + \|u(t)\|^2)\, dt$$

subject to $\dot{x}(t) = Ax(t) + Bu(t)$ on $t \leqslant 0$ for $x(0) = x_0$, and $u(\cdot) \in L_2(-\infty, 0]$ such that $x(t)$ tends to zero as $t \to -\infty$.

(β) From (30)–(31), the control $u(\cdot)$ in (32) has the feedback form

$$u(t) = -B^*(P_+ - W^\#)x(t) \quad \text{on } t \leqslant 0 \tag{36 a}$$

such that the corresponding closed-loop state trajectory (32 e), (with $x_0 \in \mathcal{R}(W)$), reads

$$x(t) = \exp([A - BB^*(P_+ - W^\#)]t)x_0 \quad \text{on } t \leqslant 0 \tag{36 b}$$

(γ) If (A) holds with (A, B) controllable, then (see Comment 1β)), $P_+ - W^\# = P_+ - W^{-1} = P_- \leqslant 0$ and (e.g. Callier and Willems 1981, p. 1234), $\mathcal{N}(P_-) = ND(C, A)$. Parts ($b$) and ($c$) above show that, if (A) holds with (A, B) not completely controllable, then P_- may be replaced by $P_+ - W^\#$ on $\mathcal{R}(W)$ for obtaining similar properties.

(δ) The subspace $\mathcal{N}(S) \cap ND(C, A)$ in (35) is important for deciding whether the solution of the RDE converges (Callier and Willems 1981, part IV). □

Proof of Lemma 4:

(a) (32 b) and (32 c) give

$$\dot{x}(t) = A_+x(t) + Bv(t) \text{ on } t \leqslant 0,\, x(0) = x_0 \tag{37}$$

where A_+ is given by (10), and the ARE can be rewritten as

$$A_+^* P_+ + P_+ A_+ + P_+ BB^* P_+ + C^* C = 0 \tag{38}$$

Now, using successively (37), (30), (32 *d*), (20) and (23), we obtain (32 *e*), where, since A_+ is exp. stable, $x(t)$ tends to zero as $t \to -\infty$. So, by (32 *c*) and (37)–(38), this results in

$$\int_{-\infty}^{0} (\|Cx(t)\|^2 + \|u(t)\|^2)\,dt = -x_0^* P_+ x_0 + \int_{-\infty}^{0} \|v(t)\|^2\,dt$$

Hence, from (32 *d*), (21) and (30), equality (32 *a*) holds.

(*b*) Equation (33) follows from the ARE, the A-invariance of $\mathcal{R}(W)$, (31), and the Lyapunov equation (22). The fact that $P_+ - W^{\#}$ is anti-stabilizing on $\mathcal{R}(W)$, (i.e. $x(t) \to 0$ as $t \to -\infty$), follows from (35), (32 *e*) and the exp. stability of A_+.

(*c*) The first equality of (34) follows from $\mathcal{N}(I - WP_+) \subset \mathcal{R}(W)$, (30) and the fact that $W^{\#} - P_+ \geq 0$ on $\mathcal{R}(W)$ (see (32 *a*)). For establishing the second equality of (34) we first prove

$$\mathcal{N}(P_+ - W^{\#}) \cap \mathcal{R}(W) \subset ND(C, A) \tag{39}$$

Let therefore $x_0 \in \mathcal{R}(W)$ such that $x_0^*(P_+ - W^{\#})x_0 = 0$. Then, from (32)

$$x(t) = \exp(At)x_0 = W \exp(-A_+^* t) W^{\#} x_0 \quad \text{on } t \leq 0 \tag{40 a}$$

and

$$Cx(t) = C \exp(At)x_0 = 0 \quad \text{on } t \leq 0 \tag{40 b}$$

Since A_+ is exp. stable, (40 *a*) implies that $\lim_{t \to -\infty} \exp(At)x_0 = 0$, i.e. $x_0 \in L^+(A)$. Moreover, by (40 *b*), $x_0 \in NO(C, A)$. Therefore $x_0 \in ND(C, A)$. Hence (39) holds. The proof of (34) is now complete if

$$ND(C, A) \subset \mathcal{N}(P_+ - W^{\#}) \cap \mathcal{R}(W) \tag{41}$$

For this purpose, observe that $ND(C, A) \subset \mathcal{R}(W) = C(A, B)$ because (A, B) is stabilizable by assumption (A). Hence, from (33), for every $x_0 \in ND(C, A)$

$$\frac{d}{dt}[x_0^* \exp(A^* t)(P_+ - W^{\#}) \exp(At)x_0] = \|B^*(P_+ - W^{\#}) \exp(At)x_0\|^2$$

on $t \leq 0$. Now, (e.g. Callier and Willems 1981, p. 1233), from assumption (A), $x_0 \in L^+(A)$, from which, using (32 *a*)

$$0 \geq x_0^*(P_+ - W^{\#})x_0 = \int_{-\infty}^{0} \|B^*(P_+ - W^{\#}) \exp(At)x_0\|^2\,dt$$

Hence $x_0 \in \mathcal{N}(P_+ - W^{\#}) \cap \mathcal{R}(W)$ and (41) holds.

(*d*) Because of (34), (35) holds if

$$\mathcal{N}(I + W(S - P_+)) = \mathcal{N}(S) \cap \mathcal{N}(P_+ - W^{\#}) \cap \mathcal{R}(W)$$

Now the right-to-left inclusion is easy using (30). For the converse note that $\mathcal{N}(I + W(S - P_+)) \subset \mathcal{R}(W)$ and then use (30), and the fact that $S \geq 0$ and $W^{\#} - P_+ \geq 0$ on $\mathcal{R}(W)$. □

Lemmas 3 and 4 now give the following connection of criterion (26) with the literature.

Theorem 2—Convergence to P_+: *Let* (A) *hold. Let* $S = S^* \geq 0$ *be given. Then*

$$\Delta P(\tau) = P(\tau) - P_+ \to 0 \quad as\ \tau \to \infty \tag{25}$$

iff any one of the following three equivalent conditions holds

(*a*)

$$I + W(S - P_+)\ \text{is non-singular} \tag{26}$$

(*b*)

$$\mathcal{N}(S) \cap ND(C, A) = \{0\} \tag{42}$$

(*c*)

$$S - (P_+ - W^{\#}) > 0\ \text{on}\ \mathcal{R}(W) \tag{43}$$

Comments 4:

(α) Condition (42) is exactly condition (32) of Callier and Willems (1981, p. 1238) and condition (43) reduces in the controllable case to $S - P_- > 0$, a well-known criterion for attraction to P_+ (e.g. Faurre *et al.* 1979, § 5.2).

(β) (Callier and Willems 1981, p. 1239) Theorem 2 above can also be used to confirm attraction towards any positive semi-definite solution P_0 of the ARE when $\mathcal{N}(S) \cap ND(C, A)$ is A-invariant. Indeed, then the attraction reduces to the attraction of a reduced RDE towards the stabilizing solution of a corresponding reduced ARE. Observe moreover that, from Willems and Callier (1983, Theorem 4), the A-invariance of $\mathcal{N}(S) \cap ND(C, A)$ is often required as a criterion for the equivalence of large finite-horizon and infinite-horizon LQ-optimal control problems.

(γ) Proposition 2 holds if (A) holds and $S = S^* \geq 0$ satisfies (42). □

Theorem 2 and Lemma 4 lead finally to the following result.

Corollary 1—Convergence to P_+ for every $S = S^* \geq 0$: *Let* (A) *hold. Then* $\Delta P(\tau) = P(\tau) - P_+$ *converges to* 0 *as* $\tau \to \infty$ *for every* $S = S^* \geq 0$ *iff* (C, A) *is detectable, or equivalently* $I - WP_+$ *is non-singular.*

Remark: Corollary 1 shows that, under assumption (A), the attraction towards P_+ for every $S = S^* \geq 0$ holds iff the stronger assumption (B) is satisfied; this, of course, was to be expected. From now on we study properties of the convergence to P_+ under the conditions of Theorem 2. □

4. Exponential convergence of the solution of the RDE

We display first exponential convergence with computable characteristics. We start by explaining the behaviour of the sliding terminal condition $\tilde{S}(\tau)$ of $\Delta P(\tau)$ given by (24 *b*) as mentioned in Comment 2(γ).

Lemma 5—Sliding terminal condition: *Let* (A) *hold and let* $S = S^* \geq 0$ *satisfy (42). Then the symmetric matrix function* $\tilde{S}(\tau)$ *of Theorem 1, namely*

$$\tau \mapsto \tilde{S}(\tau) := (S - P_+)[I + W(\tau)(S - P_+)]^{-1} \tag{24 b}$$

is well defined on $\mathbb{R}_+$ and has the following properties:

(*a*)

$$\tau \mapsto \tilde{S}(\tau) \text{ is decreasing on } \mathbb{R}_+ \tag{44}$$

and

$$\lim_{\tau\to\infty} \tilde{S}(\tau) = \tilde{S} := (S - P_+)[I + W(S - P_+)]^{-1} \tag{45}$$

whence

$$\tilde{S}(\infty) = \tilde{S} \leqslant \tilde{S}(\tau) \leqslant \tilde{S}(0) = S - P_+ \quad \text{on } \tau \geqslant 0 \tag{46}$$

(*b*)

$$\tau \mapsto \tilde{S}(\tau) \text{ is bounded on } \mathbb{R}_+; \tag{47}$$

more precisely

$$K(S) := \sup\{\|\tilde{S}(\tau)\|: \tau \geqslant 0\} \tag{48 a}$$

satisfies

$$K(S) = \max(\|S - P_+\|, \|\tilde{S}\|) \tag{48 b}$$

Proof:

(*a*) Property (44) follows because the derivative of $\tilde{S}(\tau)$ on $\tau \geqslant 0$ is negative semi-definite. To see this, observe that from (24 *b*)

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\tilde{S}(\tau) = -\tilde{S}(\tau)\cdot\frac{\mathrm{d}}{\mathrm{d}\tau}W(\tau)\cdot\tilde{S}(\tau) \quad \text{on } \tau \geqslant 0 \tag{49}$$

where, from (20), $\mathrm{d}/\mathrm{d}\tau(W(\tau))$ is positive semi-definite. For (45), note that from assumption (A) and Theorem 2, $I + W(S - P_+)$ is non-singular. Thus, by (24 *b*) and (21), (45) holds. Equation (46) obviously follows from (44)–(45) and (20).

(*b*) Properties (47)–(48) are a straightforward consequence of (46), since, for any symmetric matrix M, $\|M\| = |\lambda|_{\max}(M) = \max(-\lambda_{\min}(M), \lambda_{\max}(M))$. □

Now, with $\tilde{S}$ defined in (45), consider $\Pi(t, t_1, \tilde{S}) = \Pi(t - t_1, 0, \tilde{S}) =: \Pi(\tau)$, (with $\tau = t_1 - t$), as the symmetric $n \times n$-matrix solution of the linear matrix differential equation

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\Pi = A_+^*\Pi + \Pi A_+ \tag{50 a}$$

with

$$\Pi(0) = \tilde{S} \tag{50 b}$$

Then

$$\Pi(\tau) = \exp(A_+^*\tau)\tilde{S}\exp(A_+\tau) \tag{51}$$

Recall now that, from Theorem 1, on $\tau \geqslant 0$

$$\Delta P(\tau) := P(\tau) - P_+ = \exp(A_+^*\tau)\tilde{S}(\tau)\exp(A_+\tau) \tag{24 a}$$

where (e.g. Callier and Desoer 1991, Chapter 7) with A_+ exp. stable, there exist constants $M \geqslant 1$ and $\sigma > 0$ such that

$$\|\exp(A_+\tau)\| \leqslant M\exp(-\sigma\tau) \quad \text{for all } \tau \geqslant 0 \tag{52}$$

Hence using also (45) we have

$$\lim_{\tau\to\infty} \exp(2\sigma\tau)\|\Delta P(\tau) - \Pi(\tau)\| = 0$$

or equivalently as $\tau \to \infty$

$$\|\Delta P(\tau) - \Pi(\tau)\| = o(\exp(-2\sigma\tau)) \tag{53}$$

i.e. asymptotically $\Delta P(\tau)$ behaves as $\Pi(\tau)$, which is a *linearized version* of $\Delta P(\tau)$. To see this observe that the latter is the solution of the so-called difference Riccati differential equation on $\tau = t_1 - t \geqslant 0$, given by

$$\left.\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\tau}\Delta P &= A_+^*\Delta P + \Delta P A_+ - \Delta P BB^*\Delta P \\ \Delta P(0) &= S - P_+; \end{aligned}\right\} \tag{DRDE}$$

then compare the DRDE with (50) where the quadratic term is missing. Note that, from (24 *a*) and (51), $\Delta P(\tau)$ and $\Pi(\tau)$ differ only by the sliding terminal condition $\tilde{S}(\tau)$ of the former: it is the decreasing convergence of the latter to $\tilde{S}$ which causes $P(\tau)$ to be attracted *exp. fast* towards P_+.

Theorem 3—Exponential convergence of the RDE: *Let* (A) *hold and let* $S = S^* \geqslant 0$ *satisfy (42). Let* $K(S)$ *be as in Lemma 5, i.e. (48), and let* $M \geqslant 1$ *and* $\sigma > 0$ *be such that (52) holds. Finally let* $\Pi(\cdot) = \Pi(\cdot)^*$ *be the solution of the linear matrix differential equation (50). Then*

(*a*) *there exists a constant* $K_p(S) > 0$ *(depending on* S*) given by*

$$K_p(S) = K(S)M^2 \tag{54 a}$$

such that

$$\|\Delta P(\tau)\| \leqslant K_p(S)\exp(-2\sigma\tau) \quad \textit{for } \tau \geqslant 0 \tag{54 b}$$

Hence, as $\tau \to \infty$, $P(\tau)$ *converges exponentially fast to* P_+.

(*b*) *Consider the invertible homographic transformation* Φ *defined by*

$$\Phi(X) := X[I - WX]^{-1} \tag{55 a}$$

for any X *in the set of* $n \times n$*-symmetric real matrices, namely* $S(n)$*, such that*

$$\det[I - WX] \neq 0 \tag{55 b}$$

Then

$$\Delta P(\tau) = \Phi(\Pi(\tau)) = \Pi(\tau)[I - W\Pi(\tau)]^{-1} \quad \textit{for all } \tau \geqslant 0 \tag{56 a}$$

or equivalently

$$\Pi(\tau) = \Phi^{-1}(\Delta P(\tau)) = [I + \Delta P(\tau)W]^{-1}\Delta P(\tau) \quad \textit{for all } \tau \geqslant 0 \tag{56 b}$$

(*c*) *There exists a constant* $K_\pi(S)$ *(depending on* S*) given by*

$$K_\pi(S) = \|\tilde{S}\|M^2 \tag{57 a}$$

such that

$$\|\Pi(\tau)\| \leqslant K_\pi(S)\exp(-2\sigma\tau) \quad \textit{for } \tau \geqslant 0 \tag{57 b}$$

Moreover

$$\|\Delta P(\tau) - \Pi(\tau)\| \leqslant K_p(S)\|W\|K_\pi(S)\exp(-4\sigma\tau) \quad \textit{for } \tau \geqslant 0 \tag{58}$$

Hence, as $\tau \to \infty$, $[\Delta P(\tau) - \Pi(\tau)]$ *converges to* 0 *exponentially* (*faster than* $\Delta P(\tau)$).

Proof:

(*a*) Inequality (54) holds by Theorem 1, Lemma 5(*b*) and (52).

(*b*) From Theorem 1, (45) and (51), for all $\tau \geqslant 0$, $\Delta P(\tau) := P(\tau) - P_+ = \exp(A_+^*\tau)\tilde{S}[I - \exp(A_+\tau)W\exp(A_+^*\tau)\tilde{S}]^{-1}\exp(A_+\tau) = \Pi(\tau)[I - W\Pi(\tau)]^{-1}$. Hence (55)–(56) holds.

(*c*) Inequality (57) is obvious in view of (51) and (52). Now, from (55)–(56), $\Delta P(\tau) - \Pi(\tau) = \Delta P(\tau)W\Pi(\tau)$. Hence (58) follows from (54) and (57). □

Comments 5:

(α) The bound $K(S)$ in (54) is given by (48 *b*). Hence, if $S \geqslant P_+$, then $K(S) = \|S - P_+\|$; if $S \leqslant P_+$, then $K(S) = \|\tilde{S}\|$, from which $K_p(S) = K_\pi(S)$.

(β) If A_+ is diagonalizable and $A_+ = U\Lambda U^{-1}$ where U is an eigenvector matrix of A_+ and Λ is its diagonal eigenvalue matrix, then M and σ in (52) can be chosen as $\sigma = \min\{|\mathrm{Re}\,\lambda| : \lambda \in \sigma(A_+)\}$ and $M = \|U\|\,\|U^{-1}\|$, (i.e. the conditioning number of U). If A_+ is not diagonalizable, then one may choose any $\sigma > 0$ which approaches $\min\{|\mathrm{Re}\,\lambda| : \lambda \in \sigma(A_+)\}$ from below.

(γ) Inequality (58) results in a sharpening of (53), namely on $\tau \geqslant 0$

$$\|\Delta P(\tau) - \Pi(\tau)\| = 0(\exp(-4\sigma\tau)) \tag{59}$$

This formula shows that $\Pi(\tau)$ becomes exp. fast the dominant part of $P(\tau) - P_+$. In addition, the relation (55)–(56) between $P(\tau) - P_+$ and its linearization $\Pi(\tau)$ shows that the latter is useful for reconstructing the former exactly.

(δ) $\Delta P(\tau)$ and $\Pi(\tau)$ can be seen as evolution operators acting on $S(n)$, denoted respectively by $\Delta P(\tau)[\cdot]$ and $\Pi(\tau)[\cdot]$. Then, by (55)–(56), (51) and (45), $\Delta P(\tau)[S - P_+] = \Phi(\Pi(\tau)[\Phi^{-1}(S - P_+)])$; or, more concisely

$$\Delta P(\tau) = \Phi \circ \Pi(\tau) \circ \Phi^{-1} \tag{60}$$

i.e. the (nonlinear) DRDE evolution operator $\Delta P(\tau)$ is homographically similar to the linear evolution operator $\Pi(\tau)$. This confirms Sorine and Winternitz (1985, Theorem 1, pp. 268–269) and Medanic (1982, Section V).

(ε) The bound $K_p(S)$ in (54) is an indicator of the nonlinear behaviour of $\Delta P(\tau)$ around the horizon t_1, i.e. for $\tau = t_1 - t \geqslant 0$ small. Indeed, by the DRDE, (50) and (45)

$$\Delta P(0) - \Pi(0) = (S - P_+)[I + W(S - P_+)]^{-1}W(S - P_+)$$

This may be large as $S - P_+ > 0$ becomes large: a stiff nonlinear behaviour of

$\Delta P(\tau)$ can occur for τ small. This is well predicted by a big constant $K(S)$ in (54) as follows from Comment 5(α). It is also consistent with the fact that for a large $S - P_+ > 0$ there is a finite escape time on $\tau = t_1 - t < 0$ near zero (Shayman 1986, Lemma 11, p. 42; Martin 1981, Proposition 2.1). □

An important benefit of Theorem 3 is the exponential attraction of the optimal state and control trajectories of Proposition 1 (solution of the finite-horizon problem (LQ)) towards those of Proposition 2 (solution of the infinite-horizon problem (LQ)$^\infty$) in any $L_p[0, t_1]$-norm, namely $\|\cdot\|_p$, (see Callier and Desoer 1991, Appendix A), as the horizon t_1 tends to infinity.

Theorem 4—Exponential attraction of the optimal state- and control trajectories: *Let* (A) *hold and let* $S = S^* \geqslant 0$ *satisfy* (42). *Let* $x(\cdot)$ *and* $u(\cdot)$ *be the optimal state- and control trajectories on* $[0, t_1]$ *of problem (LQ) specified by Proposition 1. Let* $\exp(A_+ \cdot)x_0$ *and* $-B^*P_+\exp(A_+ \cdot)x_0$ *be the optimal state- and control trajectories on* $[0, \infty)$ *of problem (LQ)*$^\infty$ *specified by Proposition 2. Recall the constants in (48), (52) and (54). Then*

(*a*) *There exist constants* $K_x(S)$ *and* $K_u(S)$ *given by*

$$K_x(S) = \|W\|K_p(S) \tag{61}$$

and

$$K_u(S) = \|B\|K_p(S)[M + \|W\|(K_p(S) + \|P_+\|)] \tag{62}$$

which are independent of $t_1 \geqslant t \geqslant 0$, *such that for all* $x_0 \in \mathbb{R}^n$, *for all* $t_1 > 0$, *for all* $t \in [0, t_1]$

$$\|x(t) - \exp(A_+t)x_0\| \leqslant K_x(S)\exp(-\sigma t_1)\exp(-\sigma(t_1 - t))\|x_0\| \tag{63}$$

and

$$\|u(t) + B^*P_+\exp(A_+t)x_0\| \leqslant K_u(S)\exp(-\sigma t_1)\exp(-\sigma(t_1 - t))\|x_0\| \tag{64}$$

Hence, the optimal state and control trajectories of problem (LQ) are squeezed inside reverse-time exponentially decreasing tubes centred at those of problem (LQ)$^\infty$.

(*b*) *In addition, for all* $x_0 \in \mathbb{R}^n$, *for all* $t_1 > 0$,

$$\|x(\cdot) - \exp(A_+ \cdot)x_0\|_p \leqslant \begin{cases} K_x(S)(p\sigma)^{-1/p}\|x_0\|\exp(-\sigma t_1) & \text{for } p \geqslant 1 \\ K_x(S)\|x_0\|\exp(-\sigma t_1) & \text{for } p = \infty \end{cases} \tag{65}$$

and

$$\|u(\cdot) + B^*P_+\exp(A_+ \cdot)x_0\|_p \leqslant \begin{cases} K_u(S)(p\sigma)^{-1/p}\|x_0\|\exp(-\sigma t_1) & \text{for } p \geqslant 1, \\ K_u(S)\|x_0\|\exp(-\sigma t_1) & \text{for } p = \infty \end{cases} \tag{66}$$

Hence the optimal state and control trajectories of problem (LQ) are exponentially attracted towards those of problem (LQ)$^\infty$ *as* $t_1 \to \infty$, *in any* $L_p[0, t_1]$*-norm for* $p \in [1, \infty]$.

Proof: Inequalities (65) and (66) are obvious in view of (63) and (64). Inequality (63) is obtained as follows. By Lemma 1, $x(t) = X(t)X(0)^{-1}x_0$, with $X(t) = \exp(-A_+(t_1 - t))[I + W(t_1 - t)(S - P_+)]$, whence

$$x(t) = \exp(A_+(t - t_1))L(t, t_1)\exp(A_+t_1)x_0 \text{ for all } t \in [0, t_1] \quad (67\,a)$$

where

$$L(t, t_1) := [I + W(t_1 - t)(S - P_+)][I + W(t_1)(S - P_+)]^{-1} \quad (67\,b)$$

with $W(\cdot)$ the reachability grammian (20). From (23), assumption (A) and Theorem 2, as $t_1 \to \infty$, the numerator and the denominator of (67 b) tend to the same non-singular matrix $[I + W(S - P_+)]$. Therefore, for any fixed $t \in [0, t_1]$, $\lim_{t_1 \to \infty} L(t, t_1) = I$, from which it is natural to write $L(t, t_1)$ as

$$L(t, t_1) = I - \exp(A_+(t_1 - t))W(t) \cdot \exp(A_+^*(t_1 - t))\tilde{S}(t_1) \quad (68)$$

where $\tilde{S}(\cdot)$ is the sliding terminal condition of Lemma 5. Moreover, by (23), the grammian $W(t) = W(t)^* \geqslant 0$ converges (as $t \to \infty$) by increasing towards the grammian W given by (21). Hence

$$\|W(t)\| \leqslant \|W\| \quad \text{for all } t \geqslant 0 \quad (69)$$

Now, by (67 a) and (68), for all $x_0 \in \mathbb{R}^n$ and for all $t_1 \geqslant t \geqslant 0$, $x(t) - \exp(A_+t)x_0 = -W(t)\exp(A_+^*(t_1 - t))\tilde{S}(t_1)\exp(A_+t_1)x_0$. So (63) follows from (69), (48), (52) and definitions (54 a), (61). The derivation of (64) is similar to that of (63). It is based on the identity

$$\begin{aligned} u(t) + B^*P_+\exp(A_+t)x_0 = -B^*\,[&\Delta P(t_1 - t)\exp(A_+t)x_0 \\ &+ \Delta P(t_1 - t)(x(t) - \exp(A_+t)x_0) \\ &+ P_+(x(t) - \exp(A_+t)x_0)] \end{aligned}$$

(see (7), Theorem 3(a) and (63)). □

Comment 6: The upper bounds in (65)–(66) may be used to estimate how well a large finite-horizon LQ-problem is approximated by an infinite-horizon LQ-problem. For instance, the induced uniform operator norm of the optimal state trajectory difference operator $x_0 \in \mathbb{R}^n \mapsto x(\cdot) - \exp(A_+\cdot)x_0 \in L_p[0, t_1]$ is less than some arbitrarily small $\varepsilon > 0$ if the horizon t_1 satisfies

$$t_1 > T_{1p}$$

with

$$T_{1p} = \begin{cases} \sigma^{-1}[\log(K_x(S)) - \log(\varepsilon)] & \text{for } p = \infty \\ T_{1\infty} - (p\sigma)^{-1}\log(p\sigma) & \text{for } p \geqslant 1 \end{cases}$$

Observe that $T_{1p} \to T_{1\infty}$ as $p \to \infty$. Moreover, if the infinite horizon optimal closed loop system stability margin σ is sufficiently large, namely $p\sigma \geqslant 1$, then $T_{1p} \leqslant T_{1\infty}$. Similar results hold for the optimal control trajectory difference. □

We end this section by giving a simple illustrative example.

Example 1 (Willems and Callier 1991, p. 251): Let $A = \mathrm{diag}[1, 2]$, $B = I_2$ and $C = 0$. Then the stabilizing solution $P_+ = P_+^* \geqslant 0$ of the ARE and the reachability grammian W, (21)–(22), are given by $P_+ = \mathrm{diag}[2, 4] = W^\# = W^{-1}$; and the closed loop A-matrix, (10), (52), is given by $A_+ = -A$, with $M = \sigma = 1$. From Theorem 2, the solution $P(\tau)$ of the RDE tends to P_+ iff S is positive definite. Thus, one can choose for example $S = \mathrm{diag}[a, b]$ with a and b positive. The sliding terminal condition $\tilde{S}(\tau)$, (24 b), is

$$\tilde{S}(\tau) = \mathrm{diag}[((a-2)^{-1} + 2^{-1}(1-\exp(-2\tau)))^{-1},$$
$$((b-4)^{-1} + 4^{-1}(1-\exp(-4\tau)))^{-1}]$$

Obviously $\tilde{S}(\tau)$ is decreasing and bounded on $\tau \geqslant 0$, with (see (48 a)),

$$\|\tilde{S}(\tau)\| \leqslant \max(|a-2|, |b-4|, 2a^{-1}|a-2|, 4b^{-1}|b-4|) = K(S);$$

this agrees with Lemma 5. In addition, from (24 a), on $\tau \geqslant 0$

$$\Delta P(\tau) = \mathrm{diag}[2(a-2)\exp(-2\tau)[2 + (a-2)(1-\exp(-2\tau))]^{-1},$$
$$4(b-4)\exp(-4\tau)[4 + (b-4)(1-\exp(-4\tau))]^{-1}]$$

hence $\|\Delta P(\tau)\| \leqslant K(S)\exp(-2\tau)$ for $\tau \geqslant 0$, as stated in Theorem 3(a). Observe that, for example, with b fixed, $K_p(S) = K(S) = a - 2 = \|S - P_+\|$ when a is sufficiently large, from which the large $K_p(S)$ is a good indicator of the nonlinear behaviour of $\Delta P(\tau)$ around the horizon t_1, i.e. for $\tau = t_1 - t \geqslant 0$ small, as stated in Comment 5(ε); moreover $K_p(S) = 2a^{-1}|a-2| = \|\tilde{S}\|$ when a is sufficiently small, whence $K_p(S)$ is a good indicator of the difficult convergence of $\Delta P(\tau)$ towards 0 (as $\tau \to \infty$) when S becomes almost singular. The conclusions of Theorem 3(b)–(c) can be checked similarly. For $t \in [0, t_1]$ and $x_0 = [1 \;\; 0]^*$, the optimal state trajectory difference is given by (see (67))

$$x(t) - \exp(A_+ t)x_0 = \left[\frac{(a-2)\exp(-2t_1)}{2 + (a-2)(1-\exp(-2t_1))}(\exp(-t) - \exp(t)), 0\right]^*$$

from which

$$\|x(\cdot) - \exp(A_+ \cdot)x_0\|_\infty = \frac{|a-2|(1-\exp(-2t_1))}{2 + (a-2)(1-\exp(-2t_1))}\exp(-t_1)$$

This shows that the asymptotic behaviour of the state trajectory difference is well predicted by Theorem 4, i.e. (65), where $K_x(S) = 2^{-1}K(S)$, for a small; however, for a large, the bound (65) is conservative. Finally, for all $p \geqslant 1$ and for all $\varepsilon = 10^{-n}$, $n = 1, 2, \ldots$, the constants in Comment 6 above satisfy

$$T_{1p} \leqslant T_{1\infty} \cong \max(\log|a-2| - \log(2), \log|b-4| - \log(2),$$
$$\log|a-2| - \log(a),$$
$$\log|b-4| - \log(b) + \log(2)) + 2{\cdot}303\, n$$

Hence, as was to be expected, the horizon t_1 should be chosen large whenever more accuracy is required or whenever $S - P_+ > 0$ becomes large or S becomes almost singular.

5. Conclusions

Several important aspects of the explicit formula (in Theorem 1) for $P(\tau) - P_+$, i.e. the difference between the solution $P(\tau)$ of the Riccati differential equation and the stabilizing positive semi-definite solution P_+ of the algebraic Riccati equation, have been displayed by using essentially first-principle arguments: namely (a) explicit conditions of attraction of $P(\tau)$ towards

P_+ which use only the knowledge of P_+ and the closed loop reachability grammian W, with a nice system theoretic interpretation (see Theorem 2 and Corollary 1); and (b) the exponential nature of the convergence of $P(\tau)$ to P_+ with computable characteristics (see Theorem 3). The final result, namely Theorem 4, is also interesting; it displays the exponential attraction of the finite horizon optimal state and control trajectories towards those of the infinite horizon problem as the horizon recedes to infinity. This result follows essentially from the fact that the difference $\Delta P(\tau) = P(\tau) - P_+$ is (homographically) similar to the solution of a linear matrix differential equation (see Theorems 1 and 3(b)), with a sliding terminal condition $\tilde{S}(\cdot)$ which is decreasing and bounded (see Lemma 5). Finally, the upper bounds in Theorem 4, i.e. (65)–(66), may be used to estimate how well a large finite horizon LQ problem is approximated by an infinite horizon LQ-problem.

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